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A note on the application of the method of strained coordinates to a problem of wave propagation in plasmas

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Abstract. An initial value problem for a system of partial differential equations relating to wave propagation in plasmas is considered. A dimensionless parameter μ, characterising the wave amplitude, is contained in the problem and satisfies $0 \leqslant \mu < 1$. When $\mu \ll 1$ the approximate technique of solution known as the method of strained coordinates is appropriate. For a certain class of restricted initial conditions an exact solution of the initial value problem is derived and a comparison of this solution and the approximate solution to leading order in μ, for this class, reveals good agreement for all μ. More generally a scaling of the independent and dependent variables can be found so as to eliminate μ from the problem and this property explains why, for the restricted class, the agreement is independent of μ.

1. Introduction

The system of quasi-linear partial differential equations

$$\frac{\partial W_i}{\partial T} + A_{ij}\frac{\partial W_j}{\partial X} = 0 \tag{1}$$

frequently arises in problems of one-dimensional propagation of slowly varying non-linear waves. The method of derivation of (1) is given by Whitham (1974) who provides many illustrations from the theory of water waves and non-linear optics. Essentially it is assumed that the basic equations of the particular problem are satisfied by variables which depend on space and time in two ways, namely via the faster oscillation associated with a uniform periodic wave motion with phase variable χ and the slow modulation whose space and time dependence is represented by X and T, the scales of which are considered to be much greater than the wavelength and period of the waves respectively. Equations (1) then emerge as conditions necessary to avoid secular terms in χ and correspond to averaging the uniform solution over a period in χ. The elements of the column vector $\boldsymbol{W}$ and the matrix $\mathbf{A}$ thus involve quantities which are constant for uniform waves but vary with X and T for the slowly varying waves.

In a previous paper (Gribben and Parkes 1977) we examined the propagation of slowly varying non-linear waves in a cold, slightly non-uniform plasma and derived the system (1) in which

$$\boldsymbol{W}(X, T) = \begin{bmatrix} \gamma \\ U \\ N \\ \kappa \end{bmatrix}, \qquad \boldsymbol{A}(\boldsymbol{W}) = \begin{bmatrix} U & 0 & 0 & 0 \\ -\frac{1}{2}N^{1/2} & U & 0 & 0 \\ 0 & N & U & 0 \\ 0 & \kappa & -\frac{1}{2}N^{-1/2} & U \end{bmatrix}.$$

The precise definition and physical significance of the dependent variables is given in Gribben and Parkes (1977). Here it is sufficient to note that essentially γ, U and κ measure the amplitude, group velocity and wavenumber of the waves respectively and N is the mean particle number density. The equations are seen to form a 'degenerate' hyperbolic system with a single family of characteristics $\mathrm{d}X/\mathrm{d}T = U$, occurring with multiplicity four. We note that the first three equations of (1) are independent of the fourth which may be solved separately for κ once γ, U and N have been determined.

We consider the solution of equations (1) as an initial value problem with initial conditions

$$\gamma = \mu\hat{\gamma}(X), \qquad U = \hat{U}(X), \qquad N = \hat{N}(X), \qquad \kappa = \hat{\kappa}(X), \tag{2}$$

imposed at $T = 0$. The positive dimensionless parameter μ is introduced here to characterise the magnitude of the wave amplitude and the theory on which the equations are based is valid for $0 \leqslant \mu < 1$. In Gribben and Parkes (1977) an exact solution of the equations was quoted for the case in which $\hat{\gamma}$ is quadratic in X and $\hat{U}$, $\hat{N}$ and $\hat{\kappa}$ are constants.

In both § 2 and § 3 below it is convenient to introduce a new set of independent variables ξ, η in place of X, T. The variable ξ is chosen so that $\xi = \text{constant}$ are the characteristic curves and $\eta = T$. The transformation thus requires

$$\frac{\partial}{\partial X} \equiv \frac{1}{X_\xi}\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial X} + U\frac{\partial}{\partial X} \equiv \frac{\partial}{\partial \eta},$$

using suffix notation for partial derivatives where convenient. Hence the characteristics are given by

$$X_\eta = U \tag{3}$$

and the equations (1) become

$$\gamma_\eta = 0, \tag{4}$$

$$U_\eta = N^{1/2}\gamma_\xi/2X_\xi, \tag{5}$$

$$(X_\xi N)_\eta = 0, \tag{6}$$

$$(X_\xi \kappa)_\eta = (N^{1/2})_\xi. \tag{7}$$

If ξ is set equal to X when $\eta = 0$ the initial conditions (2) become

$$\gamma = \mu\hat{\gamma}(\xi), \qquad U = \hat{U}(\xi), \qquad N = \hat{N}(\xi), \qquad \kappa = \hat{\kappa}(\xi) \qquad \text{when } \eta = 0. \tag{8}$$

We note immediately from (4) that the solution for γ is $\gamma = \mu\hat{\gamma}(\xi)$, implying that values of γ are propagated unchanged along the characteristics.

When $\mu \ll 1$, γ is small and the equations then appear to be susceptible to the approximate method of solution known as the strained coordinate technique. In § 2 we follow this approach through for the general initial conditions (8). For certain restricted initial conditions the use of this technique indicates the form of the exact solution which is derived in § 3 where brief remarks are made about its properties. The solution quoted in Gribben and Parkes (1977) is a particular example of this exact solution. In § 4 a comparison is made of the results derived from the strained coordinate method with those from the exact solution. It is found that even to leading order in μ the former gives good results over a considerable part of the domain of the exact solution *whatever* the value of μ. This result is examined further and an explanation given. Its importance lies in the possible use of the method of strained coordinates in this and similar problems when exact solutions are not available.

2. Application of the method of strained coordinates

If $\mu \ll 1$ we can deduce the leading order behaviour of the solution by noting that U_η is small and hence U is approximately constant along the characteristics which are themselves determined as straight lines to a first approximation from (3). Corresponding approximations to N and κ then follow from (6) and (7). This sequence of steps corresponds to a linear theory of the equations for small amplitude waves and is well known to fail at large times, however small μ ($\neq 0$) may be, essentially because there the straight line characteristics can no longer represent good approximations to the true, curved ones.

The usual method of improving linear theory by straining the linearized characteristic coordinates in a way which avoids the non-uniformity in validity has been given by Lighthill (see, for example, Whitham 1974 or Van Dyke 1975). In applying the method here we strain the linearised characteristic coordinate by assuming an expansion in μ of the form

$$X(\xi, \eta) = X^{(0)}(\xi, \eta) + \mu X^{(1)}(\xi, \eta) + \dots .$$

In addition we suppose

$$U(\xi, \eta) = U^{(0)}(\xi, \eta) + \mu U^{(1)}(\xi, \eta) + \dots$$

together with similar expansions for $N(\xi, \eta)$ and $\kappa(\xi, \eta)$.

The substitution of these expansions into the equations (3), (5), (6) and (7) yields for leading order terms the linearised solution already described, namely

$$\begin{aligned}
X^{(0)} &= \hat{U}(\xi)\eta + \xi, \\
U^{(0)} &= \hat{U}(\xi), \\
N^{(0)} &= \hat{N}(\xi)/(1 + \hat{U}'(\xi)\eta), \\
\kappa^{(0)} &= \left[\hat{\kappa}(\xi) + \frac{\partial}{\partial \xi}\left(\frac{2\hat{N}^{1/2}(\xi)}{\hat{U}'(\xi)}[(1 + \hat{U}'(\xi)\eta)^{1/2} - 1]\right)\right](1 + \hat{U}'(\xi)\eta)^{-1},
\end{aligned}$$

where $\hat{U}'(\xi) = \mathrm{d}\hat{U}/\mathrm{d}\xi$. We could now calculate the first order terms in μ in general, but it suffices here (and for the rest of this paper) to choose the special case $\hat{U}$ = constant for which the solutions take a much simpler form. Thus the above results become in this case

$$X^{(0)} = \hat{U}\eta + \xi, \qquad U^{(0)} = \hat{U}, \qquad N^{(0)} = \hat{N}(\xi), \qquad \kappa^{(0)} = \hat{\kappa}(\xi) + \eta \frac{\partial}{\partial \xi}(\hat{N}^{1/2}(\xi)),$$

and the results for the first order functions, which satisfy zero initial conditions, are

$$\begin{aligned}
X^{(1)} &= \tfrac{1}{4}\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)\eta^2, \\
U^{(1)} &= \tfrac{1}{2}\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)\eta, \\
N^{(1)} &= -\tfrac{1}{4}\hat{N}(\xi)[\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)]'\eta^2,
\end{aligned}$$

$$\kappa^{(1)} = -\tfrac{1}{4}\hat{\kappa}(\xi)[\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)]'\eta^2 - \tfrac{7}{24}[\hat{N}^{1/2}(\xi)]'N^{1/2}\xi\hat{\gamma}'(\xi)]'\eta^3$$
$$-\tfrac{1}{24}\hat{N}^{1/2}(\xi)[\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)]''\eta^3.$$

From this solution, for example, the characteristics $\xi = \text{constant}$ are parabolae in the (X, T) plane to $O(\mu)$ and are given by the equation

$$X = \hat{U}T + \xi + \tfrac{1}{4}\mu\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)T^2.$$

The variable γ is positive by definition and for the physically realistic case where $\hat{\gamma} = 0$ outside some finite range of ξ, say $0 \leqslant \xi \leqslant L$, and continuous for all ξ, $\hat{\gamma}'(\xi)$ is positive for ξ near to and greater than zero and negative for ξ near to and less than L. Consequently the characteristics emerging from the points $X = 0$ and $X = L$, which may be taken to bound the region of influence of the initial data to this order, have oppositely signed curvature and it may be verified that they intersect at a finite time T^*, where

$$T^{*2} = \frac{4L}{\mu(\hat{N}^{1/2}(0)\hat{\gamma}'(0) - \hat{N}^{1/2}(L)\hat{\gamma}'(L))}.$$

The solution breaks down at a time ($\leqslant T^*$) corresponding to the first intersection of characteristics within the region of influence.

We now consider the solution to first order corresponding to initial conditions which are restricted in the disturbance region by the constraint $[\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi)]' = -\lambda$, where λ is a positive constant. In this case $\hat{N}^{1/2}(\xi)\hat{\gamma}'(\xi) = \lambda(\tilde{\xi} - \xi)$, where $\xi = \tilde{\xi}$ is the characteristic corresponding to the peak of the γ profile. Henceforth we shall refer to this case as the restricted initial value problem. In the disturbance region the solution can be written

$$\begin{aligned} X &= \hat{U}\eta + \xi + \tfrac{1}{4}\lambda(\tilde{\xi} - \xi)\mu\eta^2, \\ U &= \hat{U} + \tfrac{1}{2}\lambda(\tilde{\xi} - \xi)\mu\eta, \\ N &= \hat{N}(\xi)(1 + \tfrac{1}{4}\lambda\mu\eta^2), \\ \kappa &= \hat{\kappa}(\xi)(1 + \tfrac{1}{4}\lambda\mu\eta^2) + \eta[\hat{N}^{1/2}(\xi)]'(1 + \tfrac{7}{24}\lambda\mu\eta^2). \end{aligned} \tag{9}$$

The characteristics in the (X, T) plane are given by

$$\xi = \frac{X - \hat{U}T - \frac{1}{4}\lambda\mu\tilde{\xi}T^2}{1 - \frac{1}{4}\lambda\mu T^2} = \text{constant}, \tag{10}$$

and they all converge to the point (X^*, T^*) at which the solution breaks down, where

$$X^* = \hat{U}T^* + \tilde{\xi}, \qquad T^* = 2(\lambda\mu)^{-1/2}. \tag{11}$$

3. An exact solution of the restricted initial value problem

In this section we consider an exact solution in the disturbance region corresponding to the restricted initial value problem described in § 2. This solution, for the particular case in which $\hat{N}$ and $\hat{\kappa}$ are constant and $\hat{\gamma}$ is parabolic, was quoted by Gribben and Parkes (1977) to illustrate the behaviour of a packet of electron waves propagating in a cold, slightly non-uniform plasma stream.

By continuing the procedure described in § 2 it is found that additional factors $\lambda\mu\eta^2$ appear in the higher order terms but the ξ dependence remains the same as in the first order terms. This suggests a trial solution in the form

$$X = \hat{U}\eta + \xi + (\tilde{\xi} - \xi)f((\lambda\mu)^{1/2}\eta), \tag{12}$$

$$U = \hat{U} + (\tilde{\xi} - \xi)(\lambda\mu)^{1/2}g((\lambda\mu)^{1/2}\eta), \tag{13}$$

$$N = \hat{N}(\xi)h((\lambda\mu)^{1/2}\eta), \tag{14}$$

where f, g and h are unknown functions satisfying $f(0) = g(0) = 0$, $h(0) = 1$. When these expressions are substituted into equations (3), (5) and (6) we obtain

$$f' = g, \qquad g' = \frac{h^{1/2}}{2(1-f)}, \qquad [(1-f)h]' = 0,$$

where the primes denote differentiation with respect to the variable $(\lambda\mu)^{1/2}\eta$. These equations can be solved by elementary methods for

$$f((\lambda\mu)^{1/2}\eta) = F(\sigma), \qquad g((\lambda\mu)^{1/2}\eta) = G(\sigma) \qquad \text{and} \qquad h[(\lambda\mu)^{1/2}\eta] = H(\sigma)$$

to yield the expressions

$$F = \sigma^2(2 - \sigma^2), \tag{15}$$

$$G = 2^{1/2}\sigma(1 - \sigma^2)^{-1/2}, \tag{16}$$

$$H = (1 - \sigma^2)^{-2}, \tag{17}$$

where the transformed time variable σ is given implicitly in terms of η by the relation

$$(\lambda\mu)^{1/2}\eta = \frac{\sigma}{2^{1/2}}(1 - \sigma^2)^{3/2} + \frac{3\sigma}{2^{3/2}}(1 - \sigma^2)^{1/2} + \frac{3}{2^{3/2}}\sin^{-1}\sigma, \tag{18}$$

satisfying $\sigma = 0$ when $\eta = 0$. The solution for κ is obtained easily from (7) and is

$$\kappa = \hat{\kappa}(\xi)H(\sigma) + [\hat{N}^{1/2}(\xi)]'(\lambda\mu)^{-1/2}J(\sigma), \tag{19}$$

where

$$J = 2^{1/2}[\sigma(1 - \sigma^2)^{1/2} + \sin^{-1}\sigma](1 - \sigma^2)^{-2}. \tag{20}$$

The general properties of the solution follow closely the description given at the end of § 2. The characteristic curves are determined from (12) and (15) as

$$\xi = \frac{X - \hat{U}T - \tilde{\xi}\sigma^2(2 - \sigma^2)}{(1 - \sigma^2)^2} = \text{constant}, \tag{21}$$

where σ is given in terms of T by (18) with $\eta = T$. The disturbance represented by the initial profile in $0 \leqslant X \leqslant L$ is confined to the finite region R of the (X, T) plane bounded by the characteristics $\xi = 0$ and $\xi = L$. From (12) the width of R at any instant is

$$(X)_{\xi=L} - (X)_{\xi=0} = L(1 - \sigma^2)^2.$$

All the characteristics in R converge to the point (X^+, T^+) where $\sigma = 1$ and the solution breaks down. From (12), (15) and (18)

$$X^+ = \hat{U}T^+ + \tilde{\xi}, \qquad T^+ = \frac{3\pi 2^{1/2}}{8}(\lambda\mu)^{-1/2}. \tag{22}$$

Also from (18), σ is a monotonic increasing function which increases from 0 to 1 as η increases from 0 to T^+. The gradient of U, which is always negative, is constant for fixed η in $0 \leq \xi \leq L$, and its magnitude increases with η until the solution breaks down at $\sigma = 1$. Along each characteristic N increases with η.

4. Discussion

The approximate solution of the restricted initial value problem given by the expressions (9) can be compared with the exact solution obtained in § 3. Thus, for example, the time at which the solution breaks down is given by the approximation (11) as

$$T^* = 2(\lambda\mu)^{-1/2},$$

compared with the true value of

$$T^+ = \frac{3\pi 2^{1/2}}{8}(\lambda\mu)^{-1/2}$$

given by (22). Hence $T^*/T^+ = 1{\cdot}200$ and first order theory in μ overestimates T^+ by only 20%. In figure 1 the pair of characteristics bounding R in the (X, T) plane as predicted by both the exact (equation (21)) and approximate (equation (10)) solutions are plotted for a typical case. The exact characteristics and their approximations are seen to agree closely over a considerable part of the domain.

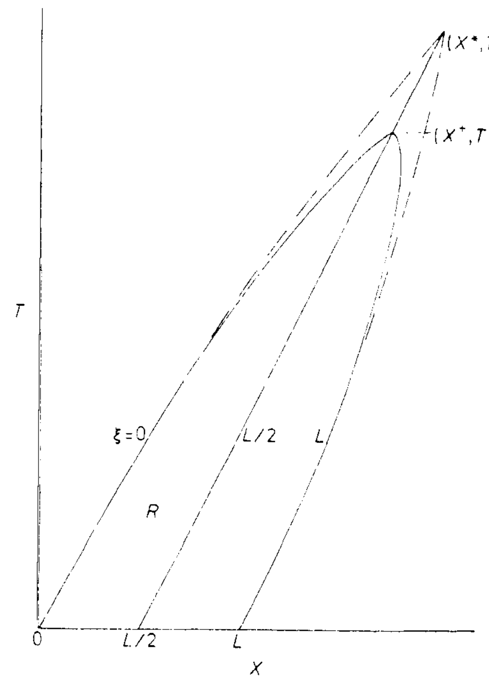

Figure 1. A diagram in the (X, T) plane of the exact and approximate (broken curves) characteristics $\xi = 0$, L, which bound the disturbed region R for the case in which $\hat{U} = \frac{1}{2}$, $L/T^* = \frac{1}{3}$, measured in appropriate velocity units and $\bar{\xi} = L/2$. The line $\xi = L/2$ corresponds to the peak of the disturbance.

To give some kind of universal comparison of approximate and exact solutions the functions F, G, H and J in (15), (16), (17) and (20) are compared in figure 2 with their approximating functions, from (9),

$$f^* = \tfrac{1}{4}\lambda\mu\eta^2, \qquad g^* = \tfrac{1}{2}(\lambda\mu)^{1/2}\eta,$$
$$h^* = 1 + \tfrac{1}{4}\lambda\mu\eta^2, \qquad j^* = (\lambda\mu)^{1/2}\eta(1 + \tfrac{7}{24}\lambda\mu\eta^2),$$

respectively, each of which depends on σ only, from (18). As the ratios f^*/F, g^*/G, h^*/H and j^*/J depend on σ only, the figures are the same whatever the values of λ and μ. It is seen that all the approximating functions are less than 10% in error over more than half the range $0 \leq T < T^+$ for which the exact solution exists. Figure 2(c) actually gives the ratio of the approximate value of N to the true value. However, to obtain the corresponding ratios for X, U and κ a particular choice of $\hat{U}$, $\hat{N}$, $\hat{\kappa}$, $\tilde{\xi}$, λ and μ must be made, as in figure 1.

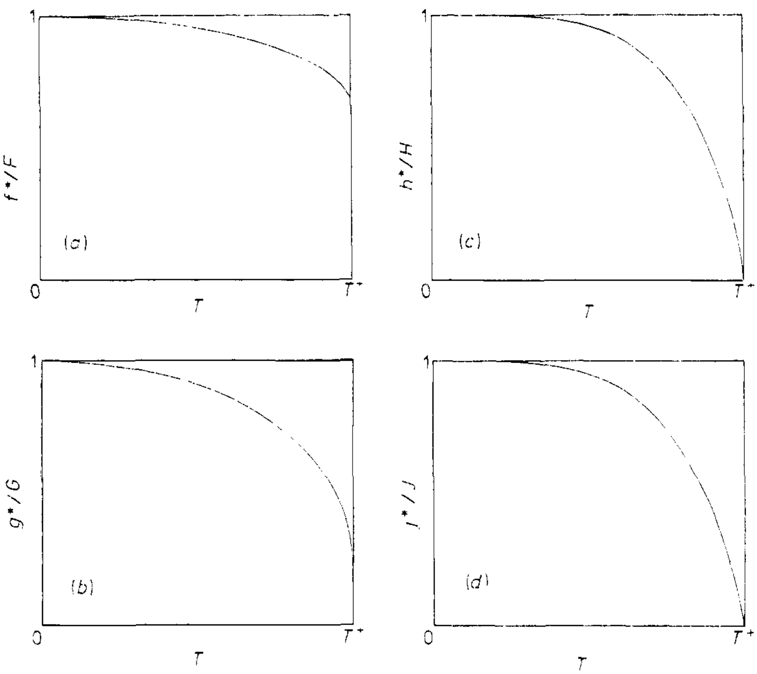

Figure 2. A comparison of the functions F, G, H and J with their approximating functions f^*, g^*, h^* and j^*.

As far as the approximate series solution obtained by the strained coordinate method is concerned, from a simple order of magnitude point of view we might expect the approximation (9) for small μ to remain a reasonable one throughout a substantial part of the domain of validity of the exact solution because the neglected terms in the series are of smaller order than the leading order terms in μ for all η of $O(\mu^{-\alpha})$, provided $\alpha < \frac{1}{2}$. At breakdown, of course, η is $O(\mu^{-1/2})$.

The comparison of the approximate solution with the exact one led to a re-examination of the basic equations. In fact, provided $\hat{U}$ is constant, the parameter μ can be eliminated from equations (3)–(7) with the otherwise more general initial conditions (8) by a stretching of the dependent and independent variables. Indeed, a clear indication of the appropriate transformation is given by the expressions (12), (13), (14), (18) and (19) used in obtaining the exact solution of the restricted initial value problem. Thus we define

$$X=\mu^{-1/2}\hat{U}\bar{\eta}+\xi+\bar{X}(\xi,\bar{\eta}),\qquad \gamma=\mu\bar{\gamma}(\xi,\bar{\eta}),\qquad U=\hat{U}+\mu^{1/2}\bar{U}(\xi,\bar{\eta}),$$
$$N=\bar{N}(\xi,\bar{\eta}),\qquad \kappa=\bar{\kappa}_1(\xi,\bar{\eta})+\mu^{-1/2}\bar{\kappa}_2(\xi,\bar{\eta}),$$

where $\bar{\eta}=\mu^{1/2}\eta$. Under this transformation the equations become

$$\begin{aligned}&\bar{X}_{\bar{\eta}}=\bar{U}, &&\bar{\gamma}_{\bar{\eta}}=0, &&\bar{U}_{\bar{\eta}}=\bar{N}^{1/2}\bar{\gamma}_\xi/2(1+\bar{X}_\xi),\\ &[(1+\bar{X}_\xi)\bar{N}]_{\bar{\eta}}=0, &&[(1+\bar{X}_\xi)\bar{\kappa}_1]_{\bar{\eta}}=0, &&[(1+\bar{X}_\xi)\bar{\kappa}_2]_{\bar{\eta}}=(\bar{N}^{1/2})_\xi\end{aligned}\tag{23}$$

satisfying the conditions

$$\begin{aligned}&\bar{X}=\bar{U}=0,\qquad \bar{N}=\hat{N}(\xi),\qquad \bar{\gamma}=\hat{\gamma}(\xi),\qquad \bar{\kappa}_1=\hat{\kappa}(\xi),\\ &\bar{\kappa}_2=0\qquad \text{on } \bar{\eta}=0.\end{aligned}\tag{24}$$

Since μ no longer appears in equations (23) and conditions (24) the method of strained coordinates is not applicable. However, a Taylor series solution in the time variable $\bar{\eta}$, with coefficients depending on ξ, can be obtained as an approximate form of solution and this would be appropriate for the initial value problem. It can be verified that such a series solution takes the form

$$\begin{aligned}&\bar{X}=\hat{N}^{1/2}\hat{\gamma}'\bar{\eta}^2/4+O(\bar{\eta}^4),\\ &\bar{U}=\hat{N}^{1/2}\hat{\gamma}'\bar{\eta}/2+O(\bar{\eta}^3),\\ &\bar{N}=\hat{N}[1-\tfrac{1}{4}(\hat{N}^{1/2}\hat{\gamma}')'\bar{\eta}^2]+O(\bar{\eta}^4),\\ &\bar{\kappa}_1=\hat{\kappa}[1-\tfrac{1}{4}(\hat{N}^{1/2}\hat{\gamma}')'\bar{\eta}^2]+O(\bar{\eta}^4),\\ &\bar{\kappa}_2=(\hat{N}^{1/2})'\bar{\eta}+O(\bar{\eta}^3),\end{aligned}\tag{25}$$

and corresponds precisely to the solution obtained by the method of strained coordinates in the original variables. Hence the latter method is of value in the present problem for general values of μ, being simply a disguised form of Taylor series solution of the differential equations. For the restricted initial value problem we have seen that the equations can be solved exactly. In other words, in this case it is possible to write down the complete sum of the Taylor series (25).

The fact that the strained coordinate method to first order in μ gives good results encourages its adoption for more general types of initial conditions in a cold plasma and for the study of the propagation of slowly varying non-linear waves in a 'warm' non-uniform plasma. In the latter case (to be reported elsewhere) the system of modulation equations is an appropriately modified version of the cold plasma equations discussed in this paper and it appears that an exact solution is not available.

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